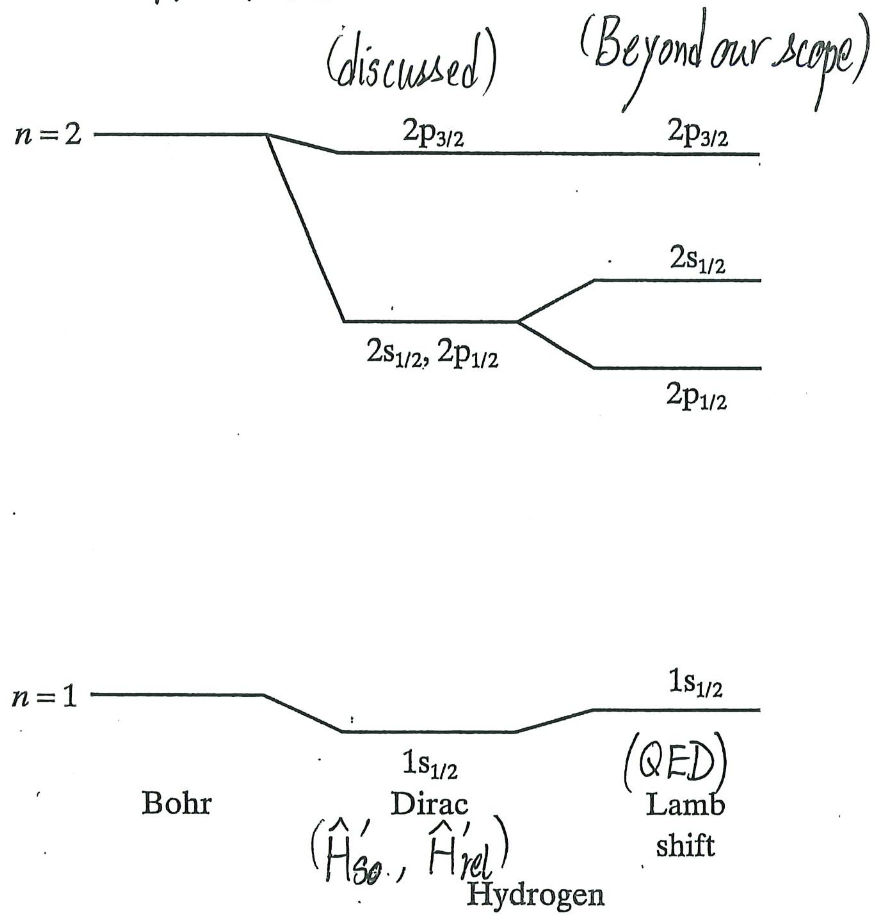
J. Hyperfine Structure

So far

(Not to scale)



[and the story continues]

As spectroscopy becomes increasingly precise, even finer details are observed.

Q: The nucleus (proton for hydrogen) has Spin Angular momentum. How does nucleus spin affect the energy levels?

Let's take stock...

- Up to now, the nucleus provides

$$U(r) = \frac{-e^2}{4\pi\varepsilon_0 r} \qquad (\text{in } \hat{H}_{atom})$$

and $U(r)$ also enters into

$$\hat{H}_{so} \sim \vec{S}\cdot(\vec{\nabla}U \times \vec{p}) \sim \frac{1}{r}\frac{dU(r)}{dr}\,\vec{S}\cdot\vec{L}$$

in spin-orbit interaction

[both related to the charge +e of the proton (nucleus)]

- But proton is a spin-½ particle of +e charge

$\Rightarrow \vec{\mu}_p$ (magnetic dipole moment) What is its effect?

Hyperfine splitting (Hydrogen 1s states) $\left(1s\ {}^2S_{1/2}\right)$

- Proton: $+e$, spin-half $s = 1/2$, $\vec{S}_p$ = spin AM of proton

Accompanying $\vec{S}_p$ is: $\vec{\mu}_p = g_p \dfrac{e}{2m_p} \vec{S}_p$

[proton mass $\approx 2000\, m_e$]

[g factor of proton, $g_p = 5.586$ experimentally]

$$\vec{\mu}_p = g_p \left(\frac{e\hbar}{2m_p}\right) \frac{1}{\hbar} \vec{S}_p \equiv g_p \, \mu_N \, \frac{1}{\hbar} \vec{S}_p$$

where μ_N = Nuclear Magneton $= \dfrac{e\hbar}{2m_p} = \underbrace{\left(\dfrac{e\hbar}{2m_e}\right)}_{\mu_B} \cdot \underbrace{\dfrac{m_e}{m_p}}_{\sim 1/2000} \approx \underbrace{3.152 \times 10^{-8}\ \text{eV/Tesla}}_{\ll \mu_B}$

(important in MRI)

$\Rightarrow$ expect effects to be tiny!

Remark:

For other nuclei, $\vec{I}$ = total spin AM of nucleus is used since there are many nucleons.

Physical Picture

Nucleus (proton) $\approx$ tiny magnet ; electron $\approx$ tiny magnet

they interact like two magnetic moments

$\vec{\mu}_N$ [proton] $(s=\frac{1}{2})$ $\qquad$ $\vec{\mu}_{spin}$ [electron] $(s=\frac{1}{2})$

$$\hat{H}'_{\substack{\text{nucleus-}\\ \text{spin}}\ \substack{\text{electron}\\ \text{spin}}} = \text{Additional interaction energy}^{+} \text{ due to nucleus spin and electron spin}$$

$$= A' \, \vec{S}_p \cdot \vec{S}_e$$

$$= A \left(\frac{\vec{S}_p}{\hbar} \cdot \frac{\vec{S}_e}{\hbar} \right) \qquad (39)$$

A: an energy giving how tiny the interaction is

$\frac{\vec{S}_p}{\hbar}$: nucleus (proton)

$\frac{\vec{S}_e}{\hbar}$: electron

Note:

- $\frac{\vec{S}}{\hbar}$ is a number

$^{+}$ Form is similar to $\hat{H}'_{so} = f(r)\, \vec{S} \cdot \vec{L}$ in spin-orbit interaction. Thus, same technique can be applied to handle $\hat{H}'_{\substack{\text{nucleus-}\\ \text{spin}}\ \substack{\text{electron}\\ \text{spin}}}$. Here, the interaction is between nucleus spin & electron spin (spin-spin). In $\hat{H}'_{so}$, the nucleus provides $V(r)$ for the electron so that $\hat{H}'_{so} \sim \vec{S} \cdot (\vec{\nabla} V \times \vec{p})$.

- Reminder on two pieces of old physics

  - See "$\vec{S}\cdot\vec{L}$", define $\vec{J} = \vec{S} + \vec{L}$ →

Recall: In spin-orbit coupling $\hat{H}'_{so} = f(r)\,\vec{S}\cdot\vec{L}$, the coupled $\vec{S}$ and $\vec{L}$ lead us to consider $\vec{J} = \vec{L} + \vec{S}$ and

$$\vec{S}\cdot\vec{L} = \frac{J^2 - L^2 - S^2}{2},$$

and states $|\ell, (s), j, m_j\rangle$ are convenient. Here, we do the same thing.

- Now, See $\vec{S}_p \cdot \vec{S}_e$, define $\vec{S}_{total} = \vec{S}_p + \vec{S}_e$

Adding two $s = 1/2$ angular momenta?

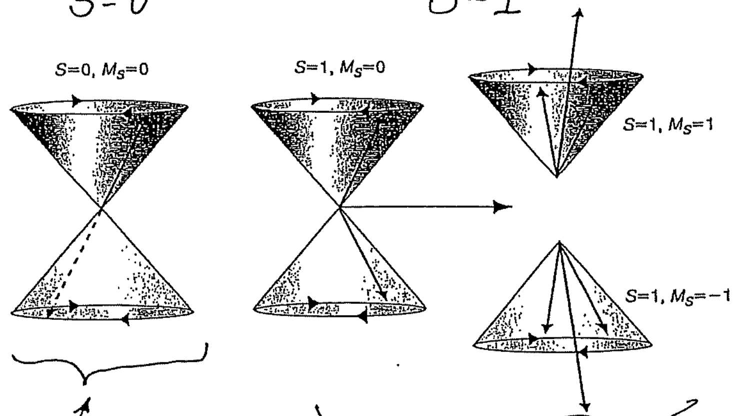
$S = 1$ (triplet)

$S = 0$ (singlet)

Here, $\vec{S}_1$ is nucleus spin
$\vec{S}_2$ is electron spin

Recall: Adding two spin-½ angular Momenta

$S = 0$ $\qquad$ $S = 1$



Vector model of the singlet and triplet states. The individual spin angular momentum vectors and their vector sum S (black arrow) are shown for the triplet states. For the singlet state (left image), $|S| = 0$ and $M_S = 0$. The dashed arrow in the left image indicates that the vector on the yellow cone is on the opposite side of the cone from the vector on the blue cone.

Singlet state: Two spin angular momenta tend to be anti-parallel

triplet states: Two spin angular momenta tend to be aligned

Let's see what happens to Hydrogen atom 1s states

$$\hat{H}'_{\text{hyperfine}} = \hat{H}'_{\substack{\text{nucleus-}\\ \text{spin}}\ \substack{\text{electron}\\ \text{spin}}} = A\left(\frac{\vec{S}_p}{\hbar}\cdot\frac{\vec{S}_e}{\hbar}\right) \propto \vec{S}_p\cdot\vec{S}_e$$

[1s : $n=1$, $l=0$, $\underset{\text{electron}}{s}=\frac{1}{2}$, $j=\frac{1}{2}$ (due to s, "$l=0$") $\left(2 \text{ states } \begin{pmatrix} m_j=\pm\frac{1}{2} \\ \text{OR} \\ m_s=\pm\frac{1}{2}\end{pmatrix} \text{ of } \underline{\text{same energy}}\right)$
(ignore $\hat{H}'_{\text{hyperfine}}$)]

Introduce: $\vec{S}_{\text{total}} = \vec{S}_p + \vec{S}_e$ ($s_p=\frac{1}{2}$, $s_e=\frac{1}{2}$ both spin-half particles)

spin quantum numbers

$$|\vec{S}_{\text{total}}| = \sqrt{S(S+1)}\,\hbar \qquad \text{with} \begin{cases} S=1, \ m_S=1,0,-1 \ \text{(triplet)} & [\vec{S}_p, \vec{S}_e \text{ tend to align}] \\ S=0, \ m_S=0 \ \text{(singlet)} & [\text{tend to anti-align}] \end{cases}$$

$$S_{\text{total},z} = m_S\hbar$$

$$\vec{S}_p\cdot\vec{S}_e = \frac{S^2_{\text{total}} - S_p^2 - S_e^2}{2} \text{ takes on } \frac{\left[S(S+1) - \frac{3}{4} - \frac{3}{4}\right]\hbar^2}{2} \text{ depending on } S$$

1s states (can use $s_e = 1/2$ and m_s OR $j = 1/2$ and m_j)
[doesn't matter because $l = 0$]

Without $\hat{H}'_{hyperfine}$, can use $\Big| \underbrace{\overset{\text{nucleus' spin}}{\overbrace{s_p, m_{s_p}}}}_{\substack{s_p = 1/2 \text{ always} \\ \text{Not invoked before}}} ; \overset{\text{"n"}}{1}, \overset{\text{"l"}}{0}, \overset{\text{"m}_l\text{"}}{0}, \underbrace{\overset{1/2 \text{ always}}{s_e}, m_s}_{\text{electron spin}} \Big\rangle$

Not invoked before [no need, nucleus effect not included]

With $\hat{H}'_{hyperfine}$, invoke $\Big| \underset{\substack{\uparrow \\ S = 1, 0}}{S}, m_S, \underset{= 1/2}{s_p}, \underset{= 1/2}{s_e}, \overset{\text{"n"}}{1}, \overset{\text{"l"}}{0}, \overset{\text{"m}_l\text{"}}{0} \Big\rangle$ is useful

Why? $\hat{H}'_{hyperfine} | S, \dots \rangle = \left[\dfrac{\overset{\text{depends on value of quantum number } S}{S(S+1)} - 3/4 - 3/4}{2} \right] A | S, \dots \rangle$

States of $S=1$: $\frac{\vec{S}_p}{\hbar}\cdot\frac{\vec{S}_e}{\hbar} = \frac{2-3/4-3/4}{2} = +1/4$
(triplet)

State of $S=0$: $\frac{\vec{S}_p}{\hbar}\cdot\frac{\vec{S}_e}{\hbar} = \frac{0-3/4-3/4}{2} = -3/4$
(singlet)

∴ 1st order perturbation†:

$$E_{hf}(\underbrace{S=1, m_s}_{\text{3 states}}) = \frac{A}{4}$$

$$E_{hf}(S=0, m_s=0) = \frac{-3A}{4}$$

} shift in energy due to hyperfine interaction

"A" here is actually same expectation value $\langle A \rangle$, c.f. $\langle f(r) \rangle$ in treating $\hat{H}'_{so}$

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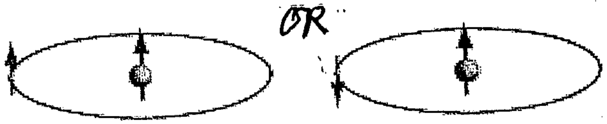
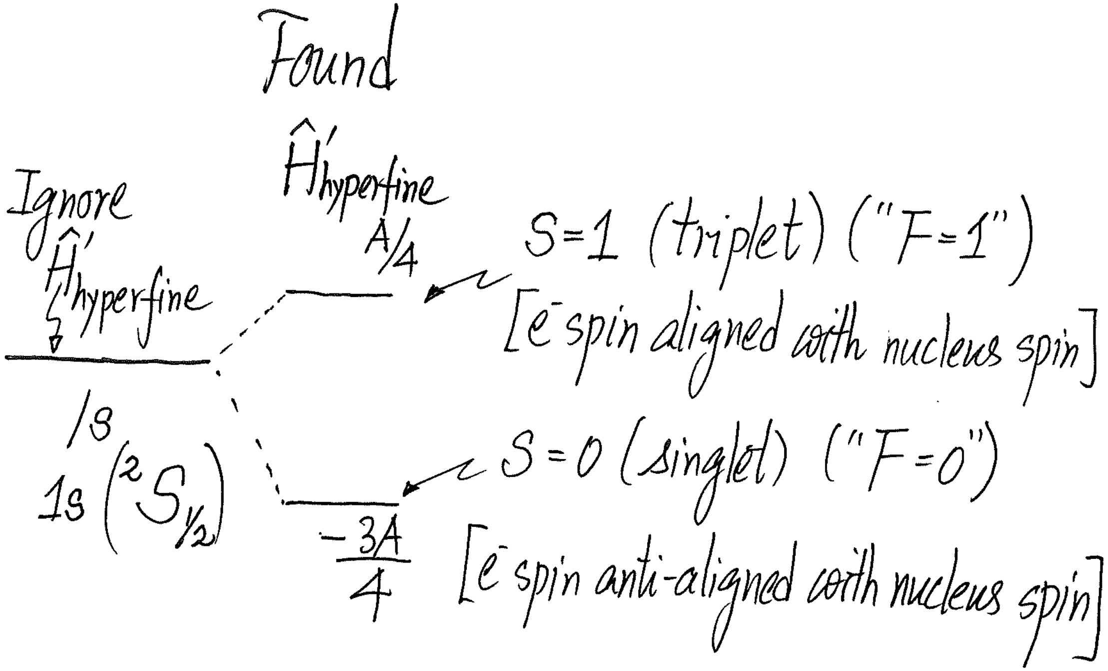
† In most books, the total spin (nucleus + electron) is labelled by the quantum number F. So, $(F=1, m_F)$ are the triplet states and $(F=0, m_F=0)$ is singlet. We avoided new notations for simplicity.

Pictorially With $\hat{H}'_{hyperfine}$

amounts to asking which of the hydrogen 1s state



has a lower energy?



"Hyperfine splitting"

How big is the hyperfine splitting in H-atom?

For hydrogen 1s:

$$\Delta E_{\text{hyperfine}} = A = \frac{4\, g_p\, \hbar^4}{m_p m_e^2 c^2 a_B^4} \cong \underbrace{5.88 \times 10^{-6}}_{\text{Note order of magnitude}}\, \text{eV}$$

The corresponding frequency is:

$$\nu = \frac{\Delta E}{h} = \underbrace{1420 \text{ MHz}}_{\text{Important! (See below)}}$$

Measured to be 1420 405 751.7667 Hz (very accurately known)

Due to works by Goldenberg, Kleppner, Ramsey

Wrote a nice textbook on Mechanics

1989 Nobel Prize

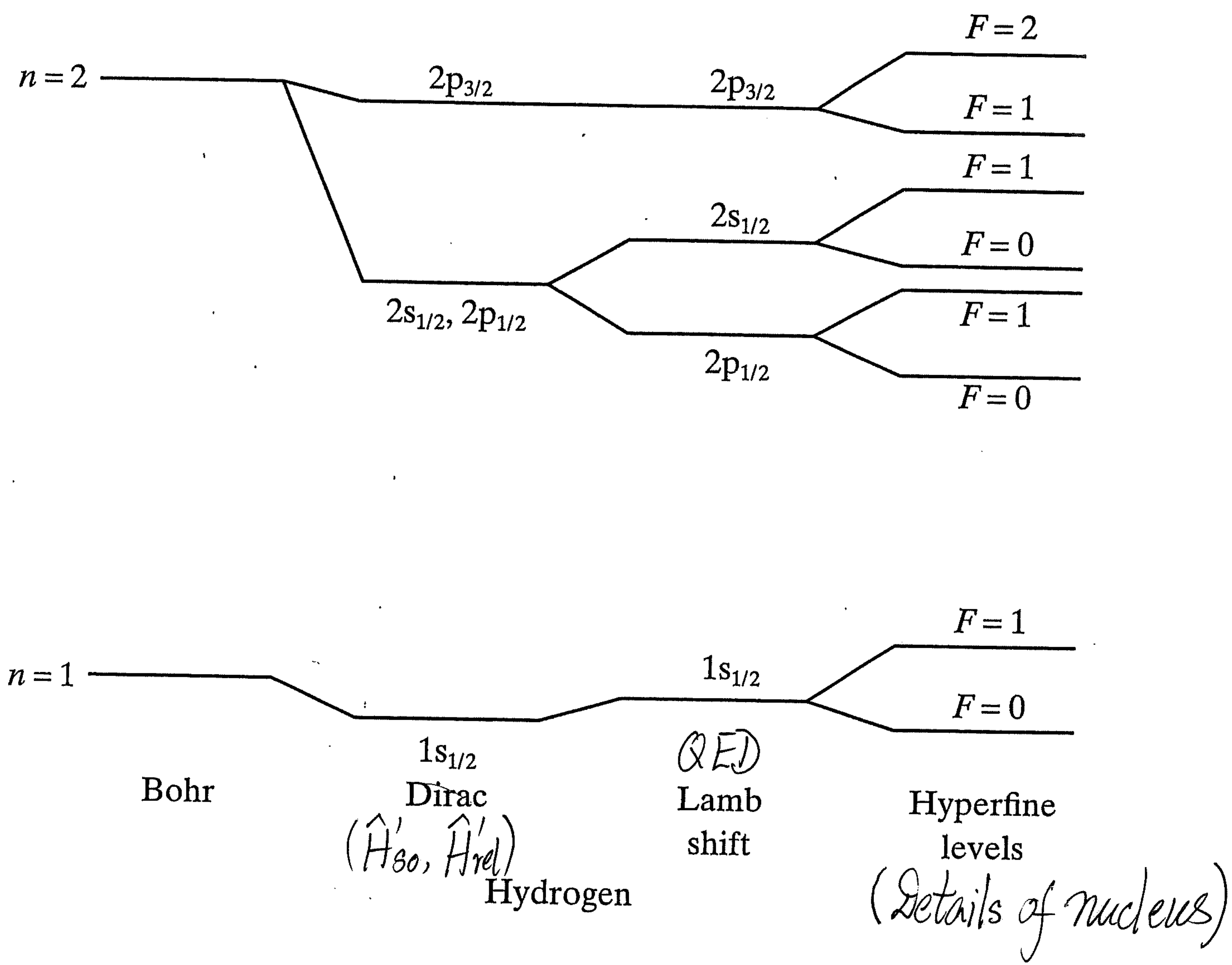
The corresponding wavelength is:

$$\lambda = \frac{c}{\nu} = \underbrace{21.121 \text{ cm}}_{\text{or the "21-cm line" of hydrogen}}$$

"21 cm cosmology"

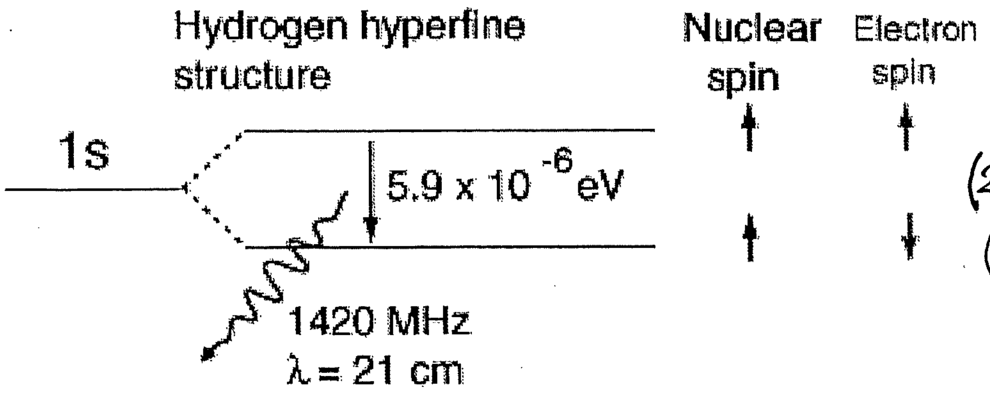
Finally, putting all effects together† (hydrogen atom)

[Not to scale]

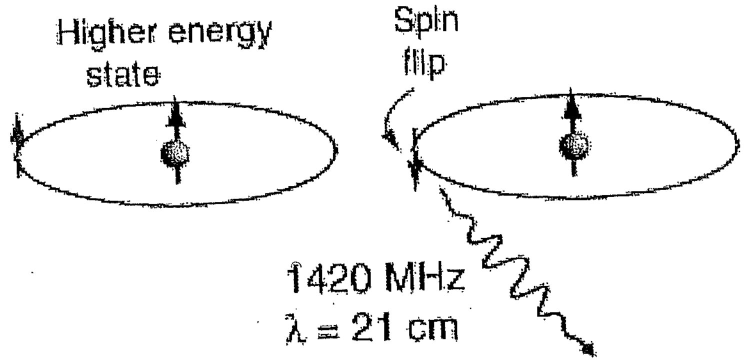


† See Remark on the notation F.

Radio Astronomy (21 cm Astrophysics)



(21.1 cm, 1420 MHz)
(~0.07 K)
"temperature"



1951 Ewen and Purcell observed 21-cm line from interstellar neutral hydrogen in our galaxy (beginning of radio astronomy). The 21-cm wave can penetrate dust clouds, thus giving a map of hydrogen.

With Doppler's shift on the line, can infer velocity of source (toward us or away from us), thus beautiful spiral pictures of galaxies.

Cosmic background (~3K) radiation is responsible for excitation across $\Delta E_{hyperfine}$.

Further Reading on Hyperfine Structure and 21cm Astronomy

A more thorough treatment of Hyperstructure and Effects due to the nucleus for hydrogen atom can be found in

B.H. Bransden & C.J. Joachain, *Physics of atoms and molecules*

[first-order perturbation theory works]

21cm Astronomy/Cosmology

Links to more information on FAST

FAST (Five-hundred-meter Aperture Spherical radio Telescope) in China (貴州洼坑) finished installation in 2016. It is the most powerful radio telescope (it started to see pulsars).

For information on the design and the scientific goals, see FAST official website http://fast.bao.ac.cn/. The following picture of FAST was taken from their site.

In September 2016, FAST was completed. Science (the magazine) carried a featured article introducing the new concepts in the telescope's design. See the article entitled "The Biggest Ear" at http://science.sciencemag.org/content/353/6307/1488 (accessible via CUHK sites).

For a professional discussion on what 21cm physics can do for 21^{st} century cosmology, read the review article

J.R. Pritchard and A. Loeb, *21 cm cosmology in the 21^{st} century*, Reports on Progress in Physics **75** (2012) 086901

http://iopscience.iop.org/article/10.1088/0034-4885/75/8/086901/pdf (from CUHK sites)

Remark: Hyperfine interaction in standard notations

- Electrons: $\vec{J}$ = Total angular momentum of electrons
- Nucleus (Many nucleons): $\vec{I}$ = Total angular momentum of $\underbrace{\text{nucleons}}_{\text{protons \& neutrons}}$

$$\hat{H}'_{hyperfine} = A\left(\frac{\vec{I}}{\hbar}\cdot\frac{\vec{J}}{\hbar}\right) \propto \vec{I}\cdot\vec{J}$$

When $\hat{H}'_{hyperfine}$ is important, I_z & J_z are not good quantities any more.

Define: $\vec{F} = \vec{I} + \vec{J}$

$$\begin{cases} \text{Total angular momentum of nuclei AND electrons} \\ F^2 = F(F+1)\hbar^2 ; \; F_z = m_F\hbar \end{cases}$$

$$\vec{I}\cdot\vec{J} = \frac{F^2 - I^2 - J^2}{2}$$

$$\vec{I}\cdot\vec{J}\,|F, m_J\rangle = \left(\frac{F(F+1) - I(I+1) - J(J+1)}{2}\right)\hbar^2 |F, m_J\rangle$$

$(F = |I+J|, |I+J|-1, \ldots, |I-J|)$

Further Remarks

- Where does proton's (neutron's) spin come from?
  [There are quarks in it.]
- Proton is not a point particle (as $V(r) = -e^2/4\pi\varepsilon_0 r$ assumed)
  - What is the size of a proton?
  - Can high-precision (H-atom) spectroscopy (atomic physics) help determine the size of a proton (nuclear physics)?
- How about isotope effect?
  - Spectrum of H-atom vs Spectrum of D-atom

    D ↑ nucleus (1p+1n)

- How about non-spherical shape of (bigger) nuclei?

  [Shape of nuclei is a research area in nuclear physics]

The point is:

Hydrogen spectrum and high-precision spectroscopy can be a driving force of advancements in other branches of physics

Two fundamental questions remain active research topics

How big is the proton?

See the article in Science (Oct 2017) entitled “The Proton Size Revisited”
http://science.sciencemag.org/content/358/6359/39

[Experiments (all very accurate) gave different values]

How did proton get its spin?

See the article (March 2017) at https://phys.org/news/2017-03-proton.html

There are quarks and gluons (binding the quarks) inside the proton. Quarks are spin-half particles as well. How could the constituents account for the observed proton’s magnetic dipole moment? It is still an active research question. See Alexandrou *et al.* “Nucleon spin and momentum decomposition using lattice QCD simulations”, Phys. Rev. Lett. **119**, 142002 (2017). The article was discussed in a softer introduction at https://phys.org/news/2017-10-proton-puzzle.html

The point is: Not to take seemingly obvious questions for granted!